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## State price density estimation for options with dividend yields

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### Abstract

Option pricing is a challenging issue that requires the fulfilment of many assumptions. The market practice for the pricing of illiquid options is often based on the usage of implied volatilities of liquid options for the construction of a so-called volatility surface. Since the surface is obtained by interpolation and a smoothing procedure, it might break the no-arbitrage condition of positive state price densities or price relations. In this paper, we extend our previous works and focus on the pricing of selected options on dividend-paying stocks traded on the German market. In particular, we construct the implied volatility surface for a large selection of combinations of time to maturity and moneyness, calculate state price densities and analyse the behaviour in different time grids.

### Keywords

Arbitrage opportunity, implied volatility, option pricing, time grid, state price density

**JEL Classification:** G17, G15, G12

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## 1. Introduction

It is already well known that options provide great opportunities for hedging and speculation. This is a natural consequence of their non-linear payoff functions, which can even contain discontinuities and other tailor-made features to respond to particular fears or beliefs of market participants. Obviously, the pricing of options is even more challenging than their usage.

Functional financial markets require efficient tools for the correct valuation of each security. Although some financial derivatives have already become very liquid and in this sense approach primary assets, such as stocks or bonds, for which the price is determined solely by the equilibrium principle, the price of, say, illiquid or exotic options must be calculated by a suitable formula. If such a formula is used properly, the resulting prices should be arbitrage free with respect to the prices of all other related securities in the market. Under the assumption of complete markets, such prices should be unique and obviously should not depend on the risk attitude of a given market participant.

During the past decades, many studies have documented that the Nobel Prize-awarded Black–Scholes model (see Black and Scholes (1973) and Merton (1973)) itself is not particularly suitable for option pricing under most real circumstances, especially since it is based on normal distribution and does not allow for fat tails and non-symmetry in the returns. However, it also appears that alternative models that should in theory capture the real market evolution better, such as various complex Lévy models, are either too difficult for everyday usage or significantly case dependent and without frequent recalibration cannot lead to correct results or both. Some interesting findings regarding the testing of option-pricing models can be found, for example, in the study by Bates (1995). A review of various models for option pricing was provided, for example, by Haug (2006), while Cont and Tankov (2010) presented advanced models respecting many kinds of real market features.

The market practice (see Dupire (1994) or Derman and Kani (1994) for an introduction to the consequences) therefore became the use of the Black–Scholes model only to price illiquid or non-traded options but with so-called implied volatility, which is volatility obtained by inverting the same formula and

inserting the market price of an option with parameters (especially the underlying asset, time to maturity and moneyness) as close as possible to those of the valued option. Clearly, since illiquid options differ from liquid ones, the proper implied volatility can be obtained only by suitable interpolation, which, unfortunately, can lead to inefficient pricing at some points; that is, it may allow an arbitrage (riskless) profit. Therefore, a specific procedure to control the correctness of the result must be implemented in the interpolation and subsequent smoothing; see, for example, Benko et al. (2007).

In our previous research, see, for example, Kopa and Tichý (2014) and Tichý et al. (2014), we concentrated on how to extract all the necessary inputs from the quotations provided by the market, including the riskless rate, which we called implied. In this paper, we proceed further and use such data to construct the implied volatility surface for selected options on dividend-paying stocks and prove whether the results fulfil the arbitrage-free conditions.

We proceed as follows. In the first section, we define the Black–Scholes formula for dividend-paying options. Next, we show how to estimate the implied volatility using the reverse Black–Scholes formula in an arbitrage-free way. Finally, we evaluate the data of three equity options from the German market using two kinds of the time grid.

## 2. Black–Scholes option-pricing model

An option is a specific type of financial derivative. Similar to forward, futures and swap contracts, its value depends on the price of its underlying asset, which can be a stock, bond, currency or even commodity, but the function that determines the payoff at maturity is non-linear and might even be discontinuous. It is given by the fact that the option holder has the right to exercise the option in a given period of time but not the obligation. Obviously, such a right will be executed only if it is efficient, which means that the payoff is positive. It also implies that the option value cannot be negative.

At this point it is important to note that, according to the type of the right, we distinguish call (the right to buy the underlying asset) and put (the right to sell the underlying asset) options and that the right can be exercised only at the maturity time (European options)

or at any time at or prior to its maturity (American options). This implies that the prices of American options should not be lower than those of European options, though for options on non-dividend-paying stocks it holds that the prices of European options are the same as those of American options.

After various attempts to price options over several decades, Black and Scholes (1973) and Merton (1973), mostly independently, derived a formula to price an option on a (non-dividend) stock, the price of which follows a geometric Brownian motion:

$$S(t) = S(0)\exp\left((\mu - \sigma^2/2)t + \sigma\sqrt{t}Z\right), \quad (1)$$

where  $S$  is the price of the stock,  $\mu$  is its long-term return (drift),  $\sigma$  is its volatility and  $Z$  is a random term of standard normal distribution.

The arguments of Black and Scholes (1973) were based initially on the no-arbitrage principle – the price must be such that, if one creates a portfolio of an option (or options) and its underlying asset, there must be no sure profit. The pricing formula comes from the fact that a suitable combination of the option and its underlying asset exists that is riskless and thus must earn a riskless profit. The change in the value of such a portfolio within infinitesimal time can be expressed via a so-called Black–Scholes partial differential equation, the solution of which is the Black–Scholes formula for a European call option price:

$$V(\text{call}) = SN(d_+) - \exp(-r\tau)KN(d_-), \quad (2)$$

where

$$d_{\pm} = \frac{\ln(S/K) + (r \pm \sigma^2/2)\tau}{\sigma\sqrt{\tau}}, \quad (3)$$

where  $S$  is the spot price of the underlying asset,  $K$  is the strike price,  $\sigma$  is the volatility of its returns,  $r$  is the riskless rate,  $\tau$  is the time to maturity and  $N(\cdot)$  states the distribution function of the standard normal distribution.

In this paper we consider dividend-paying stocks. For such purposes the formula above must be modified for the additional return that is provided to the holder of the stock. Assuming that it can be approximated as a continuous-type return  $q$ , formulas (2–3) appear as follows:

$$V(q, \text{call}) = \exp(-q\tau)SN(d_+) - \exp(-r\tau)KN(d_-), \quad (4)$$

where

$$d_{\pm} = \frac{\ln(S/K) + (r - q \pm \sigma^2/2)\tau}{\sigma\sqrt{\tau}}. \quad (5)$$

A related formula for put options can easily be obtained by the put–call parity relation or by several simple operations, both of which lead just to a change of signs as follows:

$$V(q, \text{put}) = -\exp(-q\tau)SN(-d_+) + \exp(-r\tau)KN(1 - d_-). \quad (6)$$

### 3. Arbitrage-free market volatility estimation

We can now proceed to present a method for implied volatility surface estimation and state price density surface estimation. This method employs the Black–Scholes formula and semi-parametric local quadratic smoothing techniques (using kernel functions). Moreover, the modelling is performed in such a way as to avoid the violation of arbitrage-free conditions expressed in terms of state price density and so-called total variance (Kahalé, 2004; Fengler, 2012). The method was originally proposed by Benko et al. (2007) for options with no dividend yields. We will modify this method for the case in which the underlying asset is a dividend-paying stock. This means that our analysis uses the Black–Scholes formula modified for a dividend yield (4) instead of its basic form (2).

Consider data with  $n$  options on the same underlying asset. We assume that the underlying asset is a stock with the spot price  $S$  and dividend yield  $q$ . Each option is characterized by its price (observed on the market), strike price  $K_i$ , time to maturity  $\tau_i$  and corresponding risk-free rate  $r_i$ . Moreover, the strike price and spot price ratio is called moneyness, and we use it in the future form as future moneyness  $\kappa_i = K_i/(Se^{(r_i - q)\tau_i})$ . From this information we can compute observed implied volatility  $\tilde{\sigma}_i$  (using the Black–Scholes formula), and the goal is to estimate a ‘true’ implied volatility surface  $\sigma(\kappa, \tau)$  using local quadratic smoothing procedures with arbitrage-free conditions. The estimation is performed point by point for all reasonable choices of  $\kappa$  and  $\tau$ .

Combining all these, we estimate the implied volatility surface by solving the following optimization problem:

$$\begin{aligned} \min \sum_{\ell=1}^m \sum_{i=1}^n & [\tilde{\sigma}_i - \alpha_0(\ell) - \alpha_1(\ell)(\kappa_i - \kappa) - \\ & \alpha_2(\ell)(\tau_i - \tilde{\tau}_\ell) - \alpha_{11}(\ell)(\kappa_i - \kappa)^2 - \\ & \alpha_{12}(\ell)(\kappa_i - \kappa)(\tau_i - \tilde{\tau}_\ell) - \\ & \alpha_{22}(\ell)(\tau_i - \tilde{\tau}_\ell)^2]^2 \mathcal{K}_{\mathbf{H}}(\kappa - \kappa_i, \tilde{\tau}_\ell - \tau_i) \end{aligned} \quad (7)$$

subject to

$$\begin{aligned} \sqrt{\tilde{\tau}_\ell} \varphi(d_1(\ell)) & \left\{ \frac{1}{\kappa^2 \alpha_0(\ell) \tilde{\tau}_\ell} + \frac{2d_1(\ell)}{\kappa \alpha_0(\ell) \sqrt{\tilde{\tau}_\ell}} \alpha_1(\ell) \right. \\ & \left. + \frac{d_1(\ell)d_2(\ell)}{\alpha_0(\ell)} \alpha_1(\ell)^2 + 2\alpha_{11}(\ell) \right\} \\ & \geq 0, \\ & \ell = 1, \dots, m \end{aligned} \quad (8)$$

$$2\tilde{\tau}_\ell \alpha_0(\ell) \alpha_2(\ell) + \alpha_0(\ell)^2 > 0, \quad \ell = 1, \dots, m \quad (9)$$

$$\begin{aligned} \alpha_0(\ell)^2 \tilde{\tau}_\ell &< \alpha_0(\ell')^2 \tilde{\tau}_{\ell'}, \\ \tilde{\tau}_\ell &< \tilde{\tau}_{\ell'}, \ell, \ell' = 1, \dots, m \end{aligned} \quad (10)$$

$$d_1(\ell) = \frac{\alpha_0(\ell)^2 \tilde{\tau}_\ell / 2 - \log(\kappa)}{\alpha_0(\ell) \sqrt{\tilde{\tau}_\ell}}, \ell = 1, \dots, m \quad (11)$$

$$d_2(\ell) = d_1(\ell) - \alpha_0(\ell) \sqrt{\tilde{\tau}_\ell}, \ell = 1, \dots, m \quad (12)$$

where the minimization takes over variables  $\alpha_0(\ell)$ ,  $\alpha_1(\ell)$ ,  $\alpha_2(\ell)$ ,  $\alpha_{11}(\ell)$ ,  $\alpha_{12}(\ell)$ ,  $\alpha_{22}(\ell)$ ,  $\ell = 1, \dots, m$  and:

- $\kappa$  is the particular value of future moneyiness in which the estimation is performed;
- $\kappa_i, i = 1, \dots, n$  are observed values of future moneyiness computed as  $K_i / (S e^{(r_i - q)\tau_i})$ ;
- $\tilde{\tau}_\ell, \ell = 1, \dots, m$  are particular values of time to maturity;
- the estimated implied volatilities  $\hat{\sigma}(\kappa, \tilde{\tau}_\ell)$ ,  $\ell = 1, \dots, m$  are derived from optimal  $\alpha_0(\ell)$ ,  $\ell = 1, \dots, m$ , that is,  $\hat{\sigma}(\kappa, \tilde{\tau}_\ell) = \alpha_0(\ell)$ ,  $\ell = 1, \dots, m$ ;
- $\mathcal{K}_H(\kappa - \kappa_i, \tilde{\tau}_\ell - \tau_i)$  is a kernel function, for example the Epanechnikov kernel function, with bandwidth matrix

$$\mathbf{H} = \begin{pmatrix} h_\kappa & 0 \\ 0 & h_\tau \end{pmatrix};$$

see Benko et al. (2007) for more details; the choice of  $h_\kappa$  and  $h_\tau$  will be discussed further in the next section;

- constraints (8), (9) and (10) guarantee the positivity of the state price density, which is estimated as:

$$\begin{aligned} \sqrt{\tilde{\tau}_\ell} \varphi(d_1(\ell)) &\left\{ \frac{1}{\kappa^2 \alpha_0(\ell) \tilde{\tau}_\ell} + \frac{2d_1(\ell)}{\kappa \alpha_0(\ell) \sqrt{\tilde{\tau}_\ell}} \alpha_1(\ell) + \right. \\ &\left. \frac{d_1(\ell)d_2(\ell)}{\alpha_0(\ell)} \alpha_1(\ell)^2 + 2\alpha_{11}(\ell) \right\} \end{aligned} \quad (13)$$

where  $\varphi$  is the cumulative probability distribution function of the standard Gaussian distribution;

- beside the positivity of the state price density, the arbitrage-free conditions include (9) and (10) corresponding to the requirement of increasing the total variance, which is estimated as  $\alpha_0(\ell)^2 \tilde{\tau}_\ell$ ,  $\ell = 1, \dots, m$ .

#### 4. Results

In this section we focus on the estimation results of the implied volatility surface and state price density surface for three selected stocks from the German market, first assuming a normal calendar grid and second using an artificial calendar grid. The selected stocks are those for which the most data are available.

##### Normal calendar grid

We use as the data set all the available options on three stocks (BASF SE, Bayer AG and SAP SE) listed on the German option market on 8 December 2008. These

stocks pay dividends; therefore, we compute the dividend yields of these stocks first. Then we compute the estimation of the implied volatility surface using semi-parametric local quadratic smoothing by solving the non-linear programming problem (7)–(12) for each  $\kappa = 0.7, 0.71, \dots, 1.4$  step by step. In this section we consider only the times to maturity  $\tilde{\tau}_\ell$ ,  $\ell = 1, \dots, m$  observed on the market; that is, we take a so-called normal calendar grid. The next section will show the modifications for the artificial (regular) grid.

We focus on the most relevant part of the implied volatility surface; that is, only times to maturity shorter than two years are of interest. The reason is that there are usually not enough data for longer times to maturity; moreover, the implied volatility surface seems to be ‘stabilized’ for times to maturity longer than one year.

In Figs 1, 2 and 3, we show as representative pictures the estimation with the Epanechnikov kernel function, using bandwidth  $h_\kappa = 0.25$  for moneyiness and  $h_\tau = 1.3$  for time to maturity. The historical data (black dots) are well described by the estimated surface. The implied volatility smile (as a cut for a particular time to maturity) is clear for short times to maturity and becomes less noticeable as the time to maturity increases. While the bandwidth for time to maturity is simply given as the smallest possible to have enough data for all the estimations, the bandwidth for moneyiness can be chosen almost arbitrarily. To find the best bandwidth value for moneyiness, we produce the following estimations for the SPD for the fixed calendar bandwidth  $h_\tau = 1.3$  and various values of the moneyiness bandwidth using (13).

The computations are performed again with the Epanechnikov kernel function and with five representative bandwidths for moneyiness,  $h_\kappa = 0.10, 0.15, 0.20, 0.25, 0.30$ , and bandwidth  $h_\tau = 1.3$  for time to maturity. If the moneyiness bandwidth is smaller than 0.1, the lack of data does not allow for estimations in all the points. When increasing the moneyiness bandwidth, one can observe that the estimated SPDs are not smooth enough until  $h_\kappa = 0.25$ . Moreover, there is no evident difference between the results for  $h_\kappa = 0.25$  and those for  $h_\kappa = 0.30$ . Therefore, it makes no sense to use a moneyiness bandwidth larger than 0.25 (one would pointlessly increase the bias of the estimators), and we conclude that the best choice for the moneyiness bandwidth is  $h_\kappa = 0.25$ .

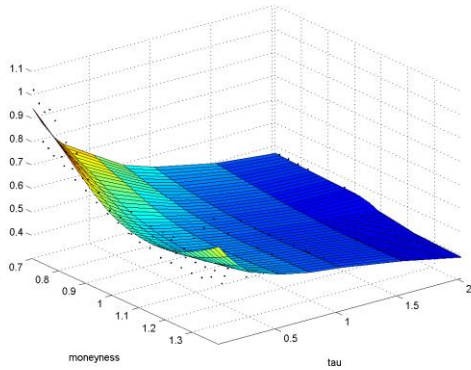


Figure 1 IV surface estimate (BASF)

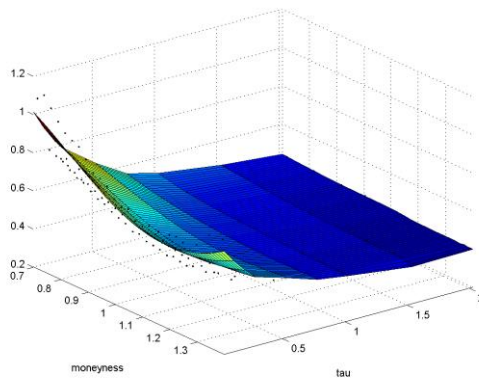


Figure 2 IV surface estimate (Bayer)

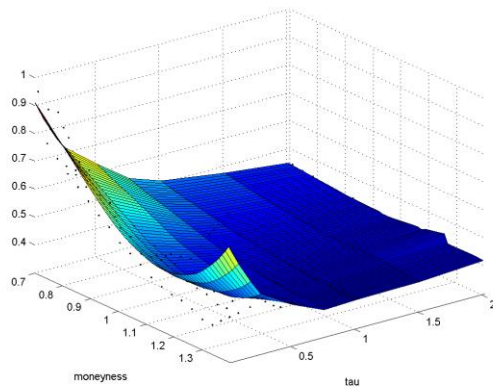


Figure 3 IV surface estimate (SAP)

We can easily check in Fig. 7 that the estimated SPD is always positive. Moreover, as shown in Fig. 8, the total variance is increasing; hence, the arbitrage-free conditions are not violated, and using (7)–(12) we obtain the arbitrage-free estimate of the implied volatility surface as proved by Benko et al. (2007).

### Artificial calendar grid

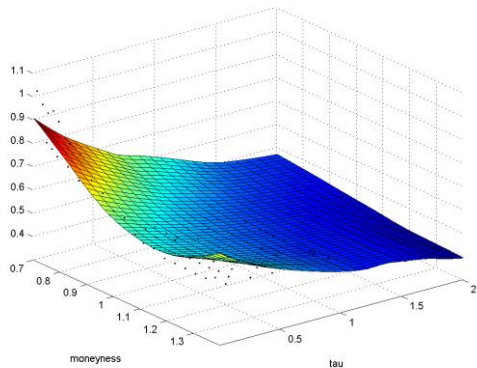
In the previous subsection, all the computations were performed using as the calendar grid the observed times to maturity of the options. In this way, the accuracy of the computations was strictly related to the data set structure. In the following analysis, contrary to Benko et al. (2007), we follow Kopa et al. (2017) and discretize the calendar grid artificially to obtain a regular grid with a one-month step. Hence, we obtain a calendar grid with twenty-four points. Moreover, we restrict the interval in the moneyness direction to avoid extreme values; that is, we again consider only  $\kappa$  from interval (0.7,1.4). Finally, we show the estimation of the implied volatility surface in Figs 4 to 6 using (7)–(12).

In Figs 4 to 6, we show as representative pictures (and for comparisons with Figs 1 to 3) the estimation with the Epanechnikov kernel function, using bandwidth  $h_\kappa = 0.25$  for moneyness and  $h_\tau = 1.3$  for time to maturity. Similarly to the previous section, using (13) we produce the estimates for the SPD (Fig. 9), concluding that the best choice for the moneyness bandwidth is again  $h_\kappa = 0.25$ .

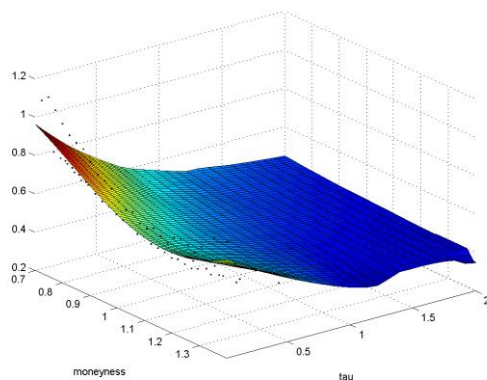
As in the previous case, the computation is performed with the Epanechnikov kernel function. Moreover, we can again easily verify that the arbitrage-free conditions are not violated. Fig. 9 shows that all the estimated SPDs are positive, and Fig. 10 shows the increasing total variances.

Finally, in Fig. 11 we propose a comparison of the estimated SPDs for the normal (observed) calendar grid with those for the artificial (regular) calendar grid for all the stocks, with bandwidth  $h_\kappa = 0.25$  for moneyness and bandwidth  $h_\tau = 1.3$  for time to maturity.

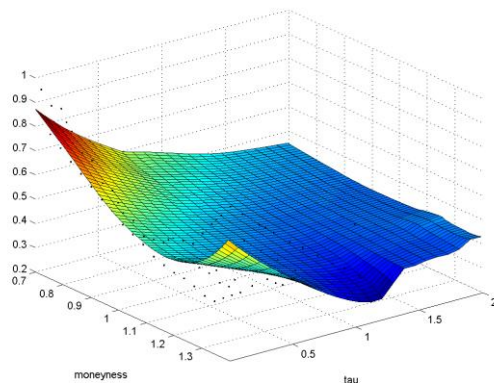
Fig. 11 shows that, using the artificial grid, one obtains more skewed state price densities for short times to maturity and less smoothed state price densities for long times to maturity. This can be the advantage of the proposed artificial grid, especially for very short (less than one month) or long (more than one year) times to maturity compared with the normal (observed) grid proposed by Benko et al. (2007).



**Figure 4** Artificial calendar grid (BASF)



**Figure 5** Artificial calendar grid (Bayer)



**Figure 6** Artificial calendar grid (SAP)

## 5. Conclusion

Although the Black–Scholes formula for option pricing cannot not be regarded as theoretically correct due to its strict assumptions on the character of the underlying asset price process, it may still be very useful for the pricing of non-traded or illiquid options for which one can find other suitable options with closely related

features that are liquid and provide the so-called implied volatilities. When accompanied by implied volatilities, it can also work as a good benchmark for more advanced models, such as complex Lévy processes.

In the paper we concentrated on developing and analysing an optimization rule, which should allow us to estimate an arbitrage-free and smooth implied volatility surface from the market prices of options on dividend-paying stocks. It was also empirically shown that state prices densities and total variances behave similarly to non-dividend-paying options when requiring an always-positive state price density and increasing total variance in maturity – both arbitrage-free conditions. Moreover, the procedure was extended to obtain a so-called artificial calendar grid, which proved to be useful especially for short- or long-lived dividend-paying options.

These results seem to be a promising step towards deeper analyses of options with discrete dividends of uncertain parameters or even American call options, for which it is well known that they might have a higher value than their sibling European calls only in the case of dividend payments.

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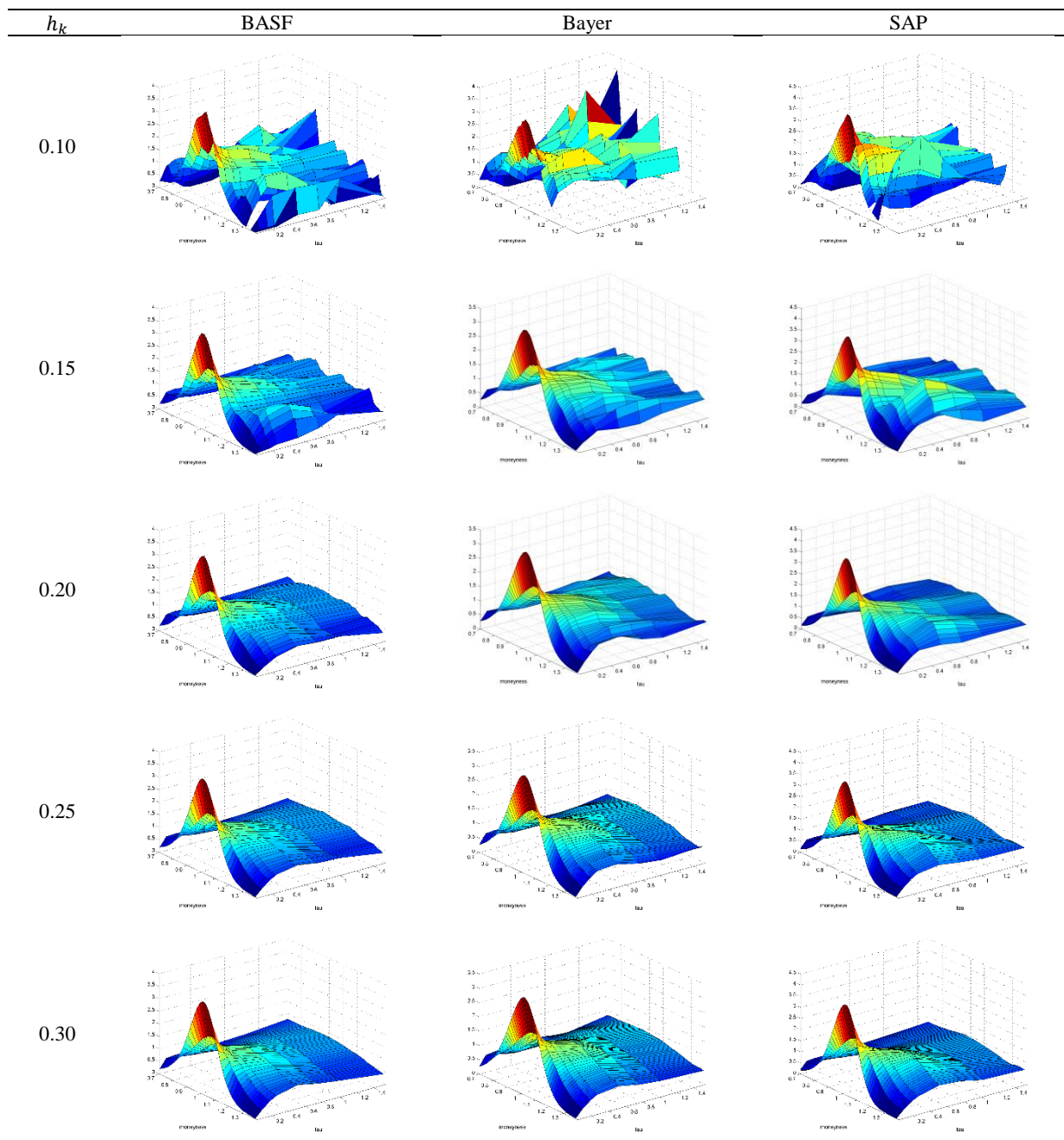
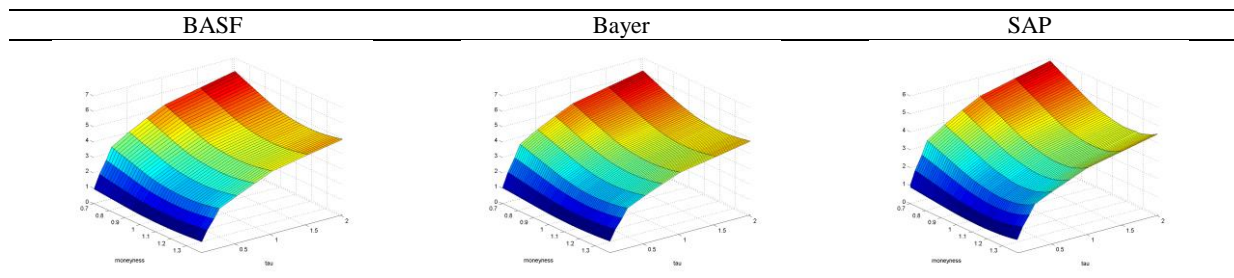
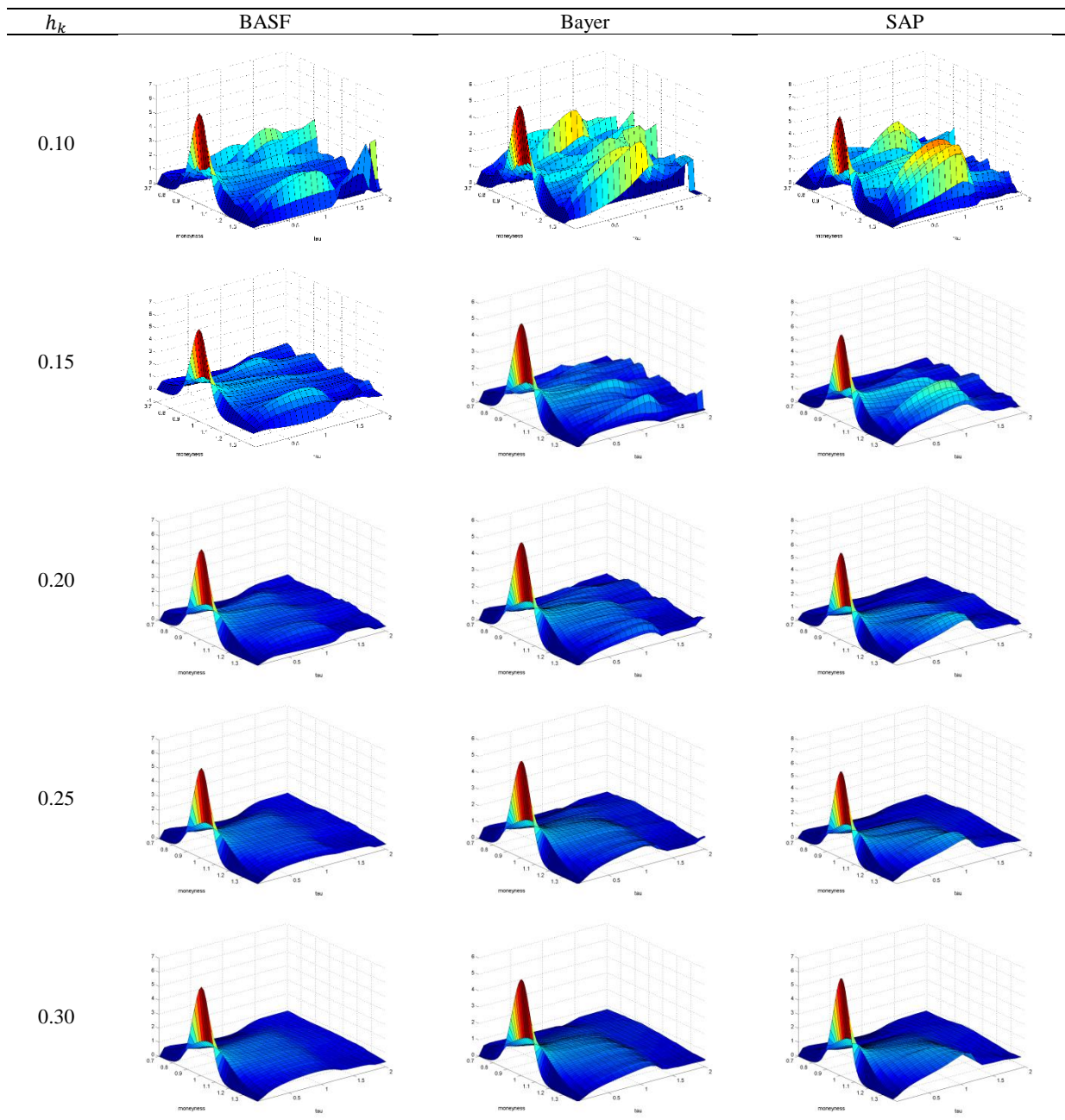


Figure 7 SPD estimate

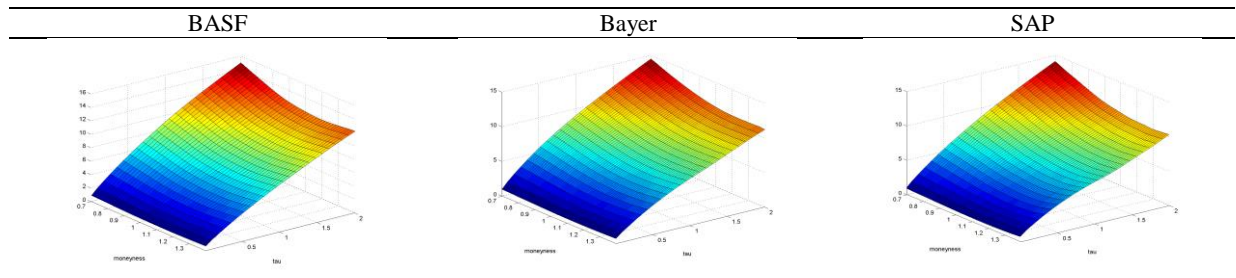


**Figure 8** Estimated total variance with normal calendar grid using  $h_k = 0.25$  and  $h_t = 1.3$ .

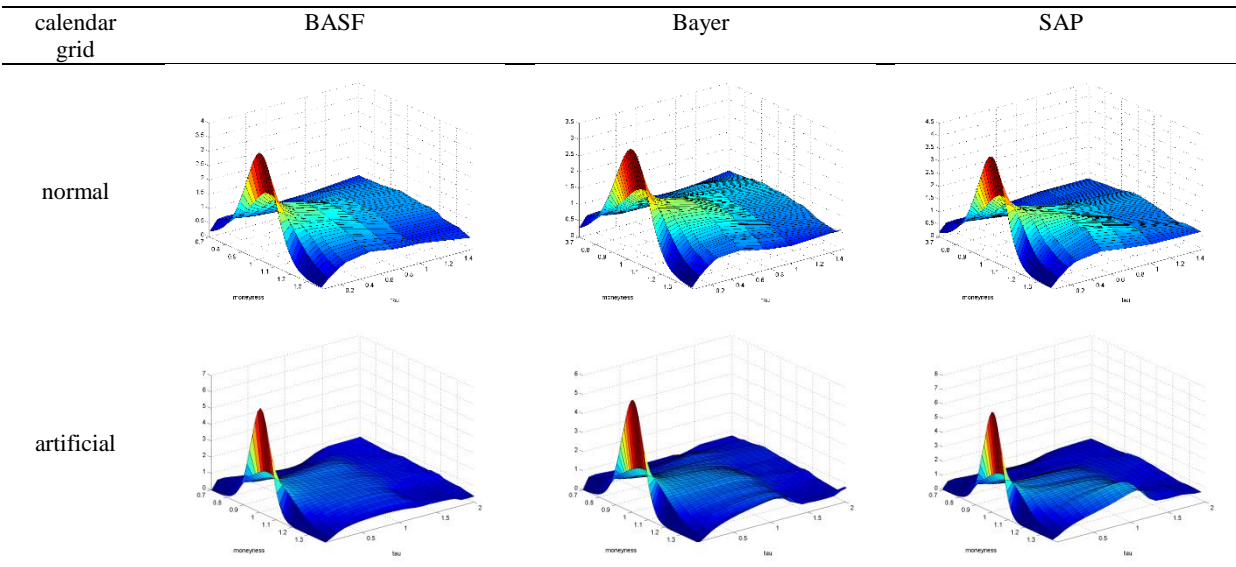


**Figure 9** SPD estimate with artificial calendar grid





**Figure 10** Estimated total variance with artificial calendar grid using  $h_{\kappa} = 0.25$  and  $h_{\tau} = 1.3$



**Figure 11** SPD estimate with normal and with artificial calendar grid

