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Numerical pricing of American options on extrema with continuous sampling

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Abstract

One of the typical option classes is formed by lookback options whose values depend also on the extrema of the underlying asset over a certain period of time. Moreover, incorporating the American constraint, which admits early exercise, has increased the popularity of these hedging and speculation instruments over recent years. In this paper, we consider the problem of pricing continuously observed American-style lookback options with fixed strike. Since no analytic formulae exist for this case, we follow an approach that formulates the corresponding option pricing problem as the parabolic partial differential inequality subject to a constraint, handled by a penalty technique. As a result, we obtain the pricing equation restricted to a triangular domain, where the path-dependent variable appears as a parameter only in the initial and boundary conditions. The contribution of the paper lies in the proposal of a numerical scheme that solves this option pricing problem. The numerical technique proposed arises from the discontinuous Galerkin that enables easy implementation of penalties and weak enforcement of boundary conditions. Finally, the capabilities of the numerical scheme are demonstrated within a simple empirical study on the reference experiments.

Keywords

option pricing; American option; lookback option; continuous sampling; Black and Scholes inequality; discontinuous Galerkin method

JEL Classification: C44, G13

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1. Introduction

The complexity of business relations leads to various cash flow patterns and often requires usage of specific securities in order to manage resulting risk properly. The most challenging among such securities are exotic options providing their owners a right to trade an underlying asset – such right is utilized only when it makes financial sense for the owner, i.e. positive cash flow is generated, and some further conditions are fulfilled, i.e. exotic options (compare with plain vanilla options).

The first rigorous treatment of option valuation dates back to the seminal papers of Black and Scholes (1973) and Merton (1973) focusing on European plain vanilla (call and put) options. Subsequent research has been focused on various extensions, such as the distribution of underlying asset price returns and their volatility, jumps consideration and also the complexity in the pay-off conditions. A detailed review of valuation models is provided by, for example, Haug (1997), while Cont and Tankov (2004) study many extensions to the underlying asset price distribution with basic pay-off classes.

While European options implicitly assume that the owner would wish to exercise his or her right just at the maturity, American options admit option exercising immediately when the owner wishes. That is why the value of American options depends on possible future decisions. It follows that the usage of numerical approximation techniques is commonly inevitable – see, for example, Duffy (2006), Topper (2005) or Hozman et al. (2018b) for their reviews.

In this contribution, we extend the previous research by Hozman and Tichý (2017) and Hozman et al. (2018a) and develop a valuation scheme for American lookback options with continuous sampling under simplifying Black and Scholes' (BS) benchmark setting, which is based on the discontinuous Galerkin (DG) method. The foundations of the DG approach are reviewed by Cockburn et al. (2005). Further, Nicholls and Sward (2015) provide American option pricing applications of the DG technique and recast the pricing problem as a linear complementarity problem. In contrast, the approach presented in this paper combines the DG method with penalty techniques proposed by Zvan et al. (1998). Concerning American lookback option pricing, authors' interest is paid in particular to binomial valuation schemes, as in Babbs (2000) and Dai (2000), or finite difference and element methods, for example, by Zhang et al. (2009) and Song et al. (2015), respectively. Unfortunately, these studies are addressed to floating strike lookback options only. Therefore, the method proposed represents a suitable alternative to these techniques that is sufficiently robust with respect to floating as well as fixed strike lookback options and option styles (European vs American). Last but not least, the advantage of this method is the discontinuous approximation that has the potential to better identify the properties of such options, when common approaches have difficulty.

The paper is organized as follows. After specifying the pricing problem for American-style options on extrema in the forthcoming section, attention is paid to the numerical pricing scheme (Section 3). Finally, in Section 4, a simple experimental study with reference results is available.

2. Options on extrema with continuous sampling

We focus on valuing a lookback option (with maturity time T) that depends on the extreme values of the underlying asset S obtained by the continuous measurement on the whole time interval $[0,t] \subset [0,T]$ as

$$m(t) = \min_{0\le\tilde{t}\le t} S(\tilde{t}) \quad \text{or} \quad M(t) = \max_{0\le\tilde{t}\le t} S(\tilde{t}) \tag{1}$$

where t is the actual time.

One can easily observe that for continuous sampling the asset price is necessarily greater than or equal to the minimum, and less than or equal to the maximum, i.e. $m \le S \le M$. This observation is not true in the case of the discrete measurement where the asset price S also takes values less than m or greater than M, cf. Wilmott et al. (1993). This fundamental difference between continuously and discretely sampled lookback options has to be reflected on the appropriate (S,m)- or (S,M)-domain on which the problem is posed.

In other words, we concentrate on fixed strike lookback options (also known as lookback rate options), for which it is characteristic that the variable m and M are incorporated into the pay-off function in the following way:

$$\begin{aligned} &\max\,(M-\mathcal{K},0), \quad \text{for a call,} \\ &\max\,(\mathcal{K}-m,0), \quad \text{for a put,} \end{aligned} \tag{2}$$

where $\mathcal{K}$ stands for a strike price. The first contract is a call option on the maximum realized price with agreed price $\mathcal{K}$, and the second one is a put option on the minimum price, respectively.

More precisely, the option value V can be viewed as a function of the actual time t, the underlying asset price $S = S(t)$ and one of the two path-dependent variables $m = m(t)$ and $M = M(t)$, respectively. The value V at maturity T is simply given by (2). In order to determine the option value V at arbitrary time instants $0 \le t < T$, the theory of semimartingales from Jacod and Shiryaev (2003) is used to characterize this value as a solution of a deterministic governing equation or inequality (according to the European or American exercise features).

Next, we introduce the BS framework modified for the specified lookback case. Suppose that the price process $S(t)$ evolves over time according to the following stochastic differential equation:

$$\mathrm{d}S(t) = \mu S(t)\mathrm{d}t + \sigma S(t)\mathrm{d}W(t), \tag{3}$$

where $\mu S(t)\mathrm{d}t$ is a drift term with a constant rate $\mu \ge 0$, $W(t)$ is a standard Brownian motion and $\sigma > 0$ is the volatility of the asset price. Unfortunately, measurements of minimum m and maximum M are not differentiable and thus have to be approximated by additional path-dependent quantities:

$$\begin{aligned} m_n(t) &= \left(\int_0^t S\,(\tilde{t})^{-n}\mathrm{d}\tilde{t}\right)^{-\frac{1}{n}}, \\ m(t) &= \lim_{n\to\infty} m_n\,(t) = \min_{0\le\tilde{t}\le t} S\,(\tilde{t}) \end{aligned} \tag{4}$$

and

$$\begin{aligned} M_n(t) &= \left(\int_0^t S\,(\tilde{t})^{n}\mathrm{d}\tilde{t}\right)^{\frac{1}{n}}, \\ M(t) &= \lim_{n\to\infty} M_n\,(t) = \max_{0\le\tilde{t}\le t} S\,(\tilde{t}). \end{aligned} \tag{5}$$

Then, the derivatives of (4) and (5) satisfy:

$$\mathrm{d}m_n(t) = -\frac{1}{n}\frac{m_n(t)^{n+1}}{S(t)^n}\,\mathrm{d}t, \tag{6}$$

$$\mathrm{d}M_n(t) = \frac{1}{n}\frac{S(t)^n}{M_n(t)^{n-1}}\,\mathrm{d}t. \tag{7}$$

Since $m(t) \le S(t)$ and $S(t) \le M(t)$ for all $t \in [0,T]$, the derivatives (6)–(7) tend to zero as $n \to \infty$. In what follows we describe both situations for European- and American-style options.

2.1 European case

The exercise of European-style options is permitted only at maturity time T. We follow the standard approach consisting of a construction of a hedged portfolio, an elimination of the stochastic components and a comparison of the portfolio dynamics by virtue of Itô's lemma. Taking these arguments into account together with vanishing terms $\mathrm{d}m_n(t)$ and $\mathrm{d}M_n(t)$ as $n \to \infty$, the price function of European lookback rate put option $V(S,m,t)$ or call option $V(S,M,t)$ can be represented as the unique solution of the partial differential equation:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0, \tag{8}$$

for $0 < m < S < \infty$ or $0 < S < M < \infty$; $0 < t \le T$, where $r \ge 0$ is the risk-free interest rate. In fact, equation (8) is the classical Black-Scholes equation in the standard variables S and t. The second spatial variable m or M enters here only as a parameter, but it also features in the terminal condition:

$$\begin{aligned} V(S,m,T) &= V_T(m), \\ V(S,M,T) &= V_T(M), \end{aligned} \tag{9}$$

where V_T is given by (2).

2.2 American case

In contrast to the European-style option, an American-style option can be exercised before the expiry of the contract. Therefore, we have to encompass the additional constraint to the problem (8)–(9) that $V \ge V_T$ at any time $t \in [0,T]$. This American feature leads to a moving-boundary problem, which consists of solving the governing equation and a determination of two regions separated by a free boundary $\mathcal{E}$ driven by the optimal exercise price S^*.

Let sets $\Omega_{\mathrm{E}}^{m} \subset \{[S,m] \in (\mathbb{R}^+ \times \mathbb{R}^+) : m < S\}$ and $\Omega_{\mathrm{E}}^{M} \subset \{[S,M] \in (\mathbb{R}^+ \times \mathbb{R}^+) : S < M\}$ denote the exercise region for a put option and call option, respectively. To unify the approach for calls and puts, we simply label Ω_{E} as the exercise region. Since it is optimal to exercise the option early in domain Ω_{E}, we solve the problem

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV < 0 \text{ in } \Omega_{\mathrm{E}} \tag{10}$$

for $0 < t \le T$, under the condition $V = V_T$.

While in the continuation region, it is not optimal to exercise early and we solve the following problem

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0 \text{ in } (\mathbb{R}^+ \times \mathbb{R}^+)\backslash\overline{\Omega_{\mathrm{E}}} \tag{11}$$

for $m < S$ or $S < M$; $0 < t \leq T$, under the condition $V > V_T$.

The well-posedness of (10)–(11) is guaranteed by a continuity of the option value V and partial derivatives $\partial V/\partial S$ and $\partial V/\partial m$ or $\partial V/\partial M$ on the free boundary $\mathcal{E}$ – see Hozman and Tichý (2020a). There are several treatments for the early-exercise feature: among those widely used let us cite the linear complementarity problem with penalty approaches by Zvan et al. (1998), or the operator splitting methods by Ikonen and Toivanen (2004). In this paper we follow the penalty technique and reformulate both problems (10) and (11) into one equation valid everywhere in both regions, i.e.

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV + q = 0, \qquad (12)$$

where $q = q(S, m, t)$ and $q = q(S, M, t)$, respectively, are defined to ensure American constraint $V \geq V_T$ and satisfy the conditions:

$$q = \begin{cases} \text{zero}, & \text{if } V > V_T, \\ \text{positive}, & \text{if } V = V_T. \end{cases} \qquad (13)$$

This new quantity q can be viewed as an additional non-linear source term in the governing equation (12) to guarantee that the value of an American option cannot fall below its pay-off function at any time t. The choice of q is specified in the next section.

3. Numerical valuation of options

Because no general analytical pricing formulae are available for finite maturity American options under BS framework, we have to rely on numerical schemes. In our study, we employ the DG method, already applied in the field of financial engineering, see, for example, Hozman and Tichý (2020a, 2018), which improves the valuation process for options. We proceed in the following steps. We start with a reformulation of the option pricing problem localized to a bounded spatial domain with forward time running. Consequently, we recall the variational form of the penalty term for the American constraint. Finally, we mention the standard spatial and temporal discretization steps and present the numerical scheme.

3.1 Initial-boundary value problem

The pricing equation (12) is accompanied by the particular pay-off (9) prescribed at maturity T. On the other hand, from the numerical point of view, it is suitable to use the forward time. Setting $\hat{t} = T - t$ the time to maturity and suppressing the dependence on S, m and M (to unify the approach), we get $u(\hat{t}) = V(t)$ and $\hat{q}(\hat{t}) = q(t)$ as a new option price function and a new penalty term, respectively.

Further, we restrict the initial problem to a bounded domain Ω. For this purpose let $S_{\max} > S^*$ and $M_{\max}$ denote the maximal sufficient value of the underlying asset and maximal possible value of its maximum, respectively. Without loss of generality $M_{\max} = S_{\max}$, i.e. we consider the upper triangular domain $\Omega := \{[S, M] \in (\mathbb{R}^+ \times \mathbb{R}^+): S < M \wedge M < M_{\max}\}$ for a lookback rate call and the lower triangular domain $\Omega := \{[S, m] \in (\mathbb{R}^+ \times \mathbb{R}^+): m < S \wedge S < S_{\max}\}$ for a lookback rate put, respectively. Consequently, the transformed governing equation with the initial condition, localized on the bounded domain, can be rewritten as

$$\frac{\partial u}{\partial \hat{t}} - \frac{\partial}{\partial S}\left(\frac{1}{2}\sigma^2 S^2 \frac{\partial u}{\partial S}\right) + \frac{\partial}{\partial S}\left((\sigma^2 - r)Su\right) + (2r - \sigma^2)u = \hat{q} \qquad (14)$$

$$u(0) = u_0 := V_T. \qquad (15)$$

Since the problem (14)–(15) is defined on the bounded domain Ω, we have to impose values of u on appropriate parts of boundary $\partial\Omega$. The prescribed values are chosen to be compatible with the pay-off function, in accordance with the vector field determined by the characteristics of (14) and by the American constraints. Since the variables m and M are not present in the differential operator in (14), the convection does not propagate in the m- and M-directions and thus no boundary condition has to be imposed on the boundary parallel to the S-axis (i.e. $m = 0$ for a lookback rate put and $M = M_{\max}$ for a lookback rate call). On line $S = 0$ the price of a lookback rate call is given by American constraint $u(0, M, \hat{t}) = M - \mathcal{K}$ that is enforced only for $M \geq \mathcal{K}$ and the remaining part $\{0\} \times (0, \mathcal{K})$ is considered as an outflow boundary. In the case of a lookback rate put, the American constraint is enforced on line $S = S_{\max}$; more precisely, we set $u(S_{\max}, m, \hat{t}) = \mathcal{K} - m$ for $m \leq \mathcal{K}$ and the remaining part $\{S_{\max}\} \times (\mathcal{K}, S_{\max})$ is considered as an outflow boundary. Finally, for the particular situation $S = m$ (put) and $S = M$ (call), we can argue that the value of the lookback option for both cases should be insensitive to infinitesimal changes in m and M, respectively, i.e. $\frac{\partial u}{\partial m}(m, m, \hat{t}) = 0$ and $\frac{\partial u}{\partial M}(M, M, \hat{t}) = 0$, see Kwok (2008).

The rigorous treatment of these boundary conditions plays an important role in achieving highly accurate solutions. A special interest here is due to the presence of a derivative boundary condition with respect to the parameter m or M. In contrast to Kwok (2008), we propose here the weak enforcement of this homogeneous boundary condition that is incorporated in the discrete formulation by the following term

$$\frac{\sigma^2}{2}\int_\Gamma S^2\left(\frac{\partial u}{\partial S}+\alpha\frac{\partial u}{\partial m}\right)n_S\,v\mathrm{d}s, \qquad (16)$$

or

$$\frac{\sigma^2}{2}\int_\Gamma S^2\left(\frac{\partial u}{\partial S}+\alpha\frac{\partial u}{\partial M}\right)n_S\,v\mathrm{d}s, \qquad (17)$$

where n_S is the first component of the outer unit vector to boundary $\Gamma = \{[S,S] \in (\mathbb{R}^+ \times \mathbb{R}^+): S < S_{\max}\}$ and $\alpha > 0$ is a suitably defined large number that represents a weight with which this boundary condition is enforced, see Hozman and Tichý (2020b).

Finally, note that relations (14)–(15) accompanied with proper boundary conditions pose the initial-boundary value problem, which is closely related to the class of convection-diffusion problems. Moreover, the governing equation (14) has no explicit dependency on the variables m and M, which are only present in the initial and boundary conditions. Therefore, the proposed numerical schemes for solving such problems have to take these properties into account.

3.2 Penalty technique

In order to handle the American early-exercise feature and force the solution of (14) to be equal to the pay-off in the exercise region Ω_{E}, we were inspired by Zvan et al. (1998) and introduce, for a sufficiently regular function v, the variational form of penalty term $\hat{q}$ as

$$\begin{aligned}&(\hat{q}(\hat{t}),v)\\ &= c_p\int_\Omega \chi_{\mathrm{exe}}(\hat{t})(u_0-u(\hat{t}))v\,\mathrm{d}S\\ &= \underbrace{c_p\int_\Omega \chi_{\mathrm{exe}}(\hat{t})u_0 v\,\mathrm{d}S}_{\mathcal{Q}_{\mathrm{R}}(v)}\\ &\quad-\underbrace{c_p\int_\Omega \chi_{\mathrm{exe}}(\hat{t})u(\hat{t})v\,\mathrm{d}S}_{\mathcal{Q}_{\mathrm{L}}(u,v)}\end{aligned} \qquad (18)$$

which where $(\cdot,\cdot)$ denotes the inner product in $L^2(\Omega)$. The function $\chi_{\mathrm{exe}}(\hat{t})$ in (18) is defined as an indicator function of the region Ω_{E} at time instant $\hat{t}$ and $c_p > 0$ represents a weight to enforce the early exercise. In line with Hozman and Tichý (2018), we set c_p proportional to $1/\tau$, where τ is the time step introduced in (21). The form (18) can be split into linear functional $\mathcal{Q}_{\mathrm{R}}$ and bilinear form $\mathcal{Q}_{\mathrm{L}}$, and we place them on opposite sides of the variational formulation of (14), see (19).

3.3 DG numerical scheme

We present a numerical scheme based on a simple modification of the DG method (see Rivière (2008) for a complete overview) that extends the lookback option pricing approach from Hozman and Tichý (2017) to the numerical pricing of American-style options using the penalty technique. The main idea of the method is to construct the solution $u_h = u_h(\hat{t})$ from the finite dimensional space S_h^p consisting of piece-wise polynomial, generally discontinuous functions of the p-th order defined over the partition $\mathcal{T}_h$ of the domain Ω with the assigned mesh size h.

The spatial discretization leads to a system of the ordinary differential equations for unknown price function u_h, i.e.

$$\begin{aligned}&\tfrac{d}{d\hat{t}}(u_h,v_h)+\mathcal{A}_h(u_h,v_h)+\mathcal{Q}_h(u_h,v_h)=\\ &\ell_h(v_h)(\hat{t})+q_h(v_h)(\hat{t}) \quad \forall\, v_h\in S_h^p,\ \forall\,\hat{t}\in(0,T),\end{aligned} \qquad (19)$$

where the initial condition $u_h(0)$ is given by (15), the bilinear forms $\mathcal{A}_h(\cdot,\cdot)$ and $\mathcal{Q}_h(\cdot,\cdot)$ stand for the discrete variants of the spatial partial differential operator from (14) and form $\mathcal{Q}_{\mathrm{L}}$ from (18), respectively. Further, the term $\ell_h(\cdot)(\hat{t})$ arises from boundary conditions and $q_h(\cdot)(\hat{t})$ is given by $\mathcal{Q}_{\mathrm{R}}$ from (18). For the detailed derivation of the above-mentioned forms we refer the interested reader to Hozman and Tichý (2020a).

Next, the temporal discretization of (19) is realized by implicit Euler method over the uniform partition of the interval $[0,T]$ with time step τ. Denote $u_h^m \in S_h^p$ the approximation of the solution $u_h(\hat{t})$ at time level $\hat{t}_m \in [0,T]$. Moreover, for practical purpose, to evaluate forms $\mathcal{Q}_h$ and q_h we use an element-wise approximation of the early exercise region as $\chi_{\mathrm{exe}}(\hat{t}_m)|_K \approx \widetilde{\chi_{\mathrm{exe}}}(\hat{t}_m)|_K$ with

$$\widetilde{\chi_{\mathrm{exe}}}(\hat{t}_m)|_K = \begin{cases}1, & \text{if } u_h^{m-1}(B_K) < u_0(B_K)\\ 0, & \text{if } u_h^{m-1}(B_K) \ge u_0(B_K)\end{cases}, \qquad (20)$$

for $\hat{t}_m \in [0,T]$, $K \in \mathcal{T}_h$, where B_K denotes a barycentre of the element K. Let $u_h^0 \approx u_0$ be the initial state, then the discrete solutions u_h^m, $m \ge 1$ are computed within the DG framework by the recurrence scheme

$$\begin{aligned}(u_h^{m+1},v_h)+\tau\mathcal{A}_h(u_h^{m+1},v_h)+\tau\mathcal{Q}_h(u_h^{m+1},v_h)&=\\ (u_h^m,v_h)+\tau\ell_h(v_h)(\hat{t}_{m+1})&\\ +\tau q_h(v_h)(\hat{t}_{m+1}) \quad \forall\, v_h\in S_h^p.&\end{aligned} \qquad (21)$$

Finally, let us mention that the equation (21) results in a sequence of systems of linear algebraic equations with non-symmetric sparse matrices. The solvability of such a system is proven in Hozman and Tichý (2018) and it is incorporated into the numerical procedure via the restarted GMRES solver.

4. Reference numerical benchmark

The experimental section introduces two numerical benchmarks widely referred to in the literature and provides the verification of the validity of the proposed numerical scheme including its capabilities. All computations are carried out with an algorithm implemented in the solver Freefem++; for more details to a mesh generation/adaptation, the spatial and temporal discretization, assembly of a linear algebraic problem, and its solving, see Hecht (2012).

Within the first experiment we examine a newly issued half-year American lookback call option with fixed strike. As in Conze and Viswanathan (1991), we consider the following model parameters: $T = 0.5$, $r = 0.1$, $\sigma = 0.2$, $\mathcal{K} \in \{95, 100, 105\}$, $S_{\mathrm{ref}} = M_{\mathrm{ref}} = 100$, $S_{\max} = M_{\max} = 2S_{\mathrm{ref}}$, where S_{ref} and M_{ref} determine the initial price and the current maximum.

For the purpose of a broader illustration of convergence properties and since the enforcement of the boundary condition on line $S = M$ is crucial for a numerical evaluation, we compute the piece-wise linear solutions on a sequence of the adaptively generated grids with the fixed number of partition nodes $n_{\mathrm{D}} \in \{125, 250, 500, 1000, 2000, 3000\}$ along line $S = M$. According to Hozman and Tichý (2020b), we take $\alpha = 500\sqrt{2}$ in (17). For all scenarios, we assume that there are 360 days in a year, take time step proportional to a quarter-day and the American early-exercise feature is handled with $c_p = 1/\tau$ in (18).

The comparative results evaluated at a given reference node $[S_{\mathrm{ref}}, M_{\mathrm{ref}}, T]$ are recorded in Table 1.1 together with bounds from Conze and Viswanathan (1991). More precisely, the following relation holds between the values of European- and American-style lookback options with fixed strike under the same market conditions

$$V_{\mathrm{Eu}} \leq V_{\mathrm{Am}} \leq V_{\mathrm{Eu}} e^{rT}, \qquad (22)$$

where V_{Eu} denotes the value of the European option and V_{Am} its American counterpart. As it is generally impossible to have closed-form expressions for the value of American options, the theoretical bounds (22) provide a suitable estimate of their prices, apart from Monte Carlo simulations or binomial method, see Babbs (2000).

Table 1.1 Comparison of American lookback rate call option values evaluated at reference node $\boldsymbol{S_{\mathrm{ref}} = M_{\mathrm{ref}} = 100}$, $\boldsymbol{\hat{t} = T}$ for different strikes and partitions.

n_{D}	$\mathcal{K} = 95$	$\mathcal{K} = 100$	$\mathcal{K} = 105$
125	20.3833	15.7586	11.4708
250	19.6407	14.8647	10.4537
500	19.2845	14.5033	10.1646
1000	19.1158	14.3132	9.9956
2000	19.0131	14.2425	9.9325
3000	18.9846	14.2163	9.9180
bounds	18.92 – 19.90	14.17 – 14.90	9.89 – 10.30

Table 1.1 is divided into three panels corresponding to the particular setting of the strike price. Unsurprisingly, one can easily observe that the proposed approach gives promising results that match tightly the range given by the reference lower and upper bounds (22) as n_{D} increases. Moreover, Figure 1.1 illustrates this behaviour for the particular grid and shows the typical findings of American-style options: that is, they cost more than their European counterparts.

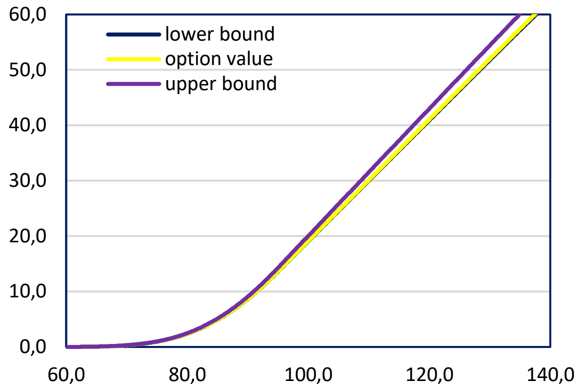

Figure 1.1 The American call option prices and the bounds related to the European counterparts at $\boldsymbol{\hat{t} = T}$ for particular setting: $\boldsymbol{\mathcal{K} = 95}$ and $\boldsymbol{n_{\mathrm{D}} = 1000}$. The horizontal axis represents the underlying asset price and the vertical one the values of options.

Secondly, we investigate the behaviour of the American lookback put option with fixed strike under the same market conditions and with the same discretization parameters as in the preceding experiment. In Table 1.2, which has the similar format to Table 1.1, we compare obtained results (evaluated at $[S_{\mathrm{ref}}, m_{\mathrm{ref}}, T]$, $m_{\mathrm{ref}} = 100$) with bounds from Conze and Viswanathan (1991). One can again observe that the obtained results are of higher accuracy and match better the reference bounds as the computational grid is finer. From this point of view, the results obtained by the DG approach are in line with conclusions from the first numerical experiment and thus the presented technique shows its promising potential in the field of numerical pricing of options.

Table 1.2 Comparison of American lookback rate put option values evaluated at reference node $\boldsymbol{S_{\mathrm{ref}} = m_{\mathrm{ref}} = 100}$, $\boldsymbol{\hat{t} = T}$ for different strikes and partitions.

n_{D}	$\mathcal{K} = 95$	$\mathcal{K} = 100$	$\mathcal{K} = 105$
125	5.7237	10.1591	14.7089
250	4.9388	9.0875	13.8720
500	4.7122	8.7042	13.5082
1000	4.5763	8.5347	13.3194
2000	4.5123	8.4480	13.2315
3000	4.4971	8.4296	13.1992
bounds	4.44 – 4.55	8.32 – 8.75	13.07 – 13.74

Finally, we also append the approximate solution captured at $\hat{t} = T$ with corresponding bounds (for particular scenario) in Figure 1.2. Similarly, it is apparent that American option prices do not fall below values of their European counterparts. Summarizing all the above mentioned, from the practical point of view, the results obtained meet the expectations of financial practitioners.

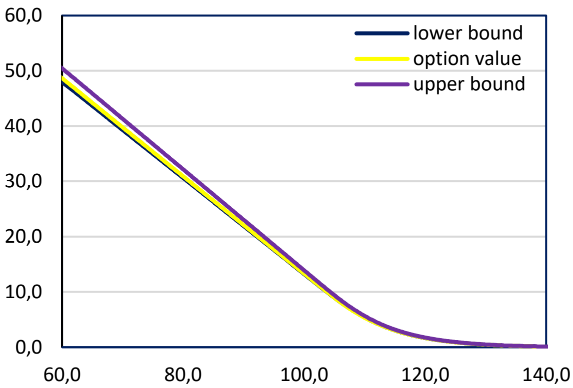

Figure 1.2 The American put option prices and the bounds related to the European counterparts at $\hat{\boldsymbol{t}} = \boldsymbol{T}$ for particular setting: $\boldsymbol{\mathcal{K}} = \mathbf{105}$ and $\boldsymbol{n}_{\mathbf{D}} = \mathbf{1000}$. The horizontal axis represents the underlying asset price and the vertical one the values of American and European put options.

5. Conclusions

Pricing of options is very challenging and a no less important part of financial engineering. In many cases, option valuation relies solely on numerical approach, as is the case of American-style options. In this contribution we have presented a numerical scheme based on the DG approach for pricing of continuously sampled American options on extrema, i.e. American lookback call and put options with fixed strike. The proposed numerical technique extends our previous results from Hozman and Tichý (2020b), where the American constraint is handled by a penalty term and the lookback feature is forced by a weak treatment of boundary conditions. The experimental study shows a quite good agreement to selected benchmarks. However, the deeper analysis is welcomed, especially concerning the sensitivity analysis and estimating the Greeks. Moreover, when the valuation procedure is combined with the approach from Hozman and Tichý (2017), one can easily price either discretely or continuously sampled lookback options. Last but not least, note that the DG method presented can be relatively easily extended to other classes of path-dependent options with different complexity of pay-off functions.

References

BABBS, S. (2000). Binomial valuation of lookback options, *Journal of Economics Dynamics & Control* 24: 1499–1525.
https://doi.org/10.1016/S0165-1889(99)00085-8

BLACK, F., SCHOLES, M. (1973). The Pricing of Options and Corporate Liabilities, *Journal of Political Economy* 81: 637–659. https://doi.org/10.1086/260062

COCKBURN, B., KARNIADAKIS, G.E., SHU, C. W., editors. (2005). *Discontinuous Galerkin Methods*. Berlin: Springer.
https://doi.org/10.1007/978-3-642-59721-3

CONT, R., TANKOV, P. (2004). *Financial modelling with jump processes*. Boca Raton: Chapman & Hall/CRC.
https://doi.org/10.1201/9780203485217

CONZE, A., VISWANATHAN, R. (1991). Path-dependent options: The case of lookback options, *Journal of Finance* 46: 1893–1907.
https://doi.org/10.1111/j.1540-6261.1991.tb04648.x

DAI, M. (2000). A Modified Binomial Tree Method for Currency Lookback Options, *Acta Mathematica Sinica* 16: 445–454.
https://doi.org/10.1007/s101140000068

DUFFY, D. J. (2006). *Finite Difference Methods in Financial Engineering: A Partial Differential Equation Approach*. Chichester: John Wiley & Sons.
https://doi.org/10.1002/9781118673447

HAUG, E. G. (1997). *The Complete Guide to Option Pricing Formulas*. 2nd edition, New York: McGraw-Hill.

HECHT, F. (2012). New development in FreeFem++, *Journal of Numerical Mathematics* 3-4: 251–265.
https://doi.org/10.1515/jnum-2012-0013

HOZMAN, J., KRESTA, A., TICHÝ, T. (2018a). Numerical Pricing of American-Style Options within the Black and Scholes Framework, *Ekonomická revue* 21: 117–123.
doi: 10.7327/cerei.2018.12.03

HOZMAN, J., TICHÝ, T. (2020a). The discontinuous Galerkin method for discretely observed Asian options, *Mathematical Methods in the Applied Sciences* 43: 7726-7746.
http://doi.org/10.1002/mma.6160

HOZMAN, J., TICHÝ, T. (2020b). Numerical Pricing of American Lookback Options with Continuous Sampling of the Maximum. In: *Mathematical Methods in Economics*. Brno: Mendel University in Brno, 186–192.

HOZMAN, J., HOLČAPEK, M., TICHÝ, T., ČERNÁ, D., KRESTA, A., VALÁŠEK, R. (2018b). *Robust Numerical Schemes for Pricing of Selected Options*, SAEI, vol. 57, Ostrava: VŠB-TUO.

HOZMAN, J., TICHÝ, T. (2018b). DG framework for pricing European options under one-factor stochastic volatility models, *Journal of Computational and Applied Mathematics* 344: 585–600.
http://doi.org/10.1016/j.cam.2018.05.064

HOZMAN, J., TICHÝ, T. (2017). The Valuation of Discretely Sampled European Lookback Options: A DG Approach. In: *Mathematical Methods in Economics*. Hradec Králové: University of Hradec Králové, 242–247.

IKONEN, S., TOIVANEN, J. (2004). Operator splitting methods for American option pricing, *Applied Mathematics Letters* 17: 809–814.
https://doi.org/10.1016/j.aml.2004.06.010

JACOD, J., SHIRYAEV, A. N. (2003). *Limit Theorems for Stochastic Processes*. Berlin: Springer.
https://doi.org/10.1007/978-3-662-05265-5

KWOK, Y. K. (2008). *Mathematical Models of Financial Derivatives*. Berlin: Springer.
https://doi.org/10.1007/978-3-540-68688-0

MERTON, R. C. (1973). Theory of Rational Option Pricing, *The Bell Journal of Economics and Management Science* 4: 141–183.
https://doi.org/10.2307/3003143

NICHOLLS, D., SWARD, A. (2015). A discontinuous Galerkin method for pricing American options under the constant elasticity of variance model, *Communications in Computational Physics* 17: 761–778.
https://doi.org/10.4208/cicp.190513.131114a

RIVIÈRE, B. (2008). *Discontinuous Galerkin Methods for Solving Elliptic and Parabolic Equations: Theory and Implementation*, Philadelphia: Society for Industrial and Applied Mathematics.
https://doi.org/10.1137/1.9780898717440

SONG, H., ZHANG, Q., ZHANG, R. (2015). A fast numerical method for the valuation of American lookback put options, *Communications in Nonlinear Science and Numerical Simulation* 27: 302–313.
https://doi.org/10.1016/j.cnsns.2015.03.010

TOPPER, J. (2005). *Financial Engineering with Finite Elements*, Chichester: John Wiley & Sons.

WILMOTT, P., DEWYNNE, J., HOWISON, J. (1993). *Option pricing: mathematical models and computations*. Oxford: Oxford financial press.

ZHANG, T., ZHANG, S., ZHU, D. (2009). Finite Difference Approximation for Pricing the American Lookback Option, *Journal of Computational Mathematics* 27: 484–494.
https://doi.org/10.4208/jcm.2009.27.4.015

ZVAN, R., FORSYTH, P.A., VETZAL, K.R (1998). Penalty methods for American options with stochastic volatility, *Journal of Computational and Applied Mathematics* 91: 199–218.
https://doi.org/10.1016/S0377-0427(98)00037-5